A PROOFS OF LEMMAS AND FACTS

A.1 Proof of Lemma 4

The proof is very similar to the proof of Lemma 2 of Heckel et al. (2018). There are several cases of q_1 and b_2 to consider. We will show each by contradiction, starting with the assumption that the termination condition is false and both $\mathcal{E}_{\text{bad}}(q_1)$ and $\mathcal{E}_{\text{bad}}(b_2)$ do not occur, all under the event $\mathcal{E}_\alpha$. Let $\mathcal{E}_{\text{good}}(i)$ denote the complement of $\mathcal{E}_{\text{bad}}(i)$. It also will be useful to define the quantity

$$m_2 = \operatorname*{arg\,max}_{i \in \{(k+1),\ldots,(k+h)\}} \alpha_i \tag{15}$$

such that $b_2 = \arg\max_{i\in\{m_2,q_2\}} \alpha_i$.

i. When $q_1 \le k$ and $b_2 > k+h$, we have by $\mathcal{E}_{\text{good}}(q_1)$ that

$$\widehat{d}_{q_1} + \alpha_{q_1} < \widehat{d}_{q_1} + 3\alpha_{q_1} \le \gamma \tag{16}$$

and similarly that $\widehat{d}_{b_2} - \alpha_{b_2} > \gamma$ by $\mathcal{E}_{\text{good}}(b_2)$. Since $\widehat{d}_{q_2} - \alpha_{q_2} \ge \widehat{d}_{m_2} - \alpha_{m_2}$, we have that $\widehat{d}_{q_2} - \alpha_{q_2} > \gamma$ in both the case that $b_2 = m_2$ and $b_2 = q_2$. Together, this implies that the termination condition (4) is true, which violates our assumption.

ii. When $q_1 \le k$ and $k < b_2 \le k+h$, we have first by $\mathcal{E}_{\text{good}}(q_1)$ that $\widehat{d}_{q_1} + 3\alpha_{q_1} \le \gamma$. Starting from here, and using the definition of q_1, we have for all $i \in \widehat{\mathcal{S}}_{\text{close}}$,

$$\begin{aligned}\gamma &\ge \widehat{d}_{q_1} + \alpha_{q_1} + 2\alpha_{q_1}\\ &\ge \widehat{d}_i + \alpha_i + 2\alpha_{q_1}\\ &\ge d_i + 2\alpha_{q_1}\\ &> d_i.\end{aligned} \tag{17}$$

Now we let Δ denote $d_{k+1+h} - d_k$. By definition of b_2, using $\mathcal{E}_{\text{good}}(b_2)$, we have that $\alpha_j \le \Delta/4$ for all $j \in \widehat{\mathcal{S}}_{\text{middle}} \cup \{q_2\}$. Then we can start from $\gamma > \widehat{d}_{q_1} + \alpha_{q_1}$ to conclude that for all $j \in \widehat{\mathcal{S}}_{\text{middle}} \cup \{q_2\}$,

$$\begin{aligned}\gamma &> \widehat{d}_{q_1} + \alpha_{q_1}\\ &\overset{(i)}{>} \widehat{d}_{q_2} - \alpha_{q_2}\\ &\ge \widehat{d}_{q_2} - \frac{\Delta}{4}\\ &\ge \widehat{d}_j - \frac{\Delta}{4}\\ &\ge d_j - \alpha_j - \frac{\Delta}{4}\\ &\ge d_j - \frac{\Delta}{2},\end{aligned} \tag{18}$$

where (i) comes from our assumption that the terminating condition (4) is false. Combining (17) and (18) along with $\gamma + \Delta/2 = d_{k+1+h}$, we obtain that $d_{k+1+h} > d_i$ for all $i \in \widehat{\mathcal{S}} \cup \{q_2\}$, which is a contradiction, since there can be at most $k+h$ values of d_i that are smaller than d_{k+1+h}.

iii. When $k < q_1 \le k+h$ and $b_2 > k+h$, the case is similar to the previous case, except that we need to bound α_i for $i \in \widehat{\mathcal{S}}_{\text{middle}}$ in a different way. By $\mathcal{E}_{\text{good}}(b_2)$, $\widehat{d}_{q_2} \ge \widehat{d}_{b_2}$, and $\alpha_{b_2} \ge \alpha_{q_2}$, we have analogously to (17), for all $i \in \widehat{\mathcal{S}}_{\text{far}}$,

$$\begin{aligned}\gamma &\le \widehat{d}_{b_2} - 3\alpha_{b_2}\\ &\le \widehat{d}_{q_2} - \alpha_{q_2} - 2\alpha_{b_2}\\ &\le d_i - 2\alpha_{b_2}.\end{aligned} \tag{19}$$

Equivalently, $d_i \ge \gamma + 2\alpha_{b_2}$. Since there are $n-k-h$ values of i for which this inequality holds, it must hold for d_{k+1+h}, so we obtain

$$\alpha_{b_2} \le \frac{d_{k+1+h} - \gamma}{2} = \frac{\Delta}{4}. \tag{20}$$

By definition of b_2, $\alpha_i \le \Delta/4$ for all $i \in \widehat{\mathcal{S}}_{\text{middle}} \cup \{q_2\}$, and a contradiction can be reached similarly as in case ii.

iv. For the case when both $q_1, b_2 \in \{k+1,\ldots,k+h\}$, we first show that at least one of $\gamma < \widehat{d}_{q_1} + \alpha_{q_1}$ or $\gamma > \widehat{d}_{q_2} - \alpha_{q_2}$ is true. To see this, first suppose the former is false. Then using that the terminating condition (4) is false, we have

$$\gamma \ge \widehat{d}_{q_1} + \alpha_{q_1} > \widehat{d}_{q_2} - \alpha_{q_2}. \tag{21}$$

Now that we know that at least one of these inequalities holds, and we proceed similarly for each. First suppose that the former inequality, $\gamma < \widehat{d}_{q_1} + \alpha_{q_1}$, holds. Using that by $\mathcal{E}_{\text{good}}(q_1)$ and $\mathcal{E}_{\text{good}}(b_2)$ we have $\alpha_i \le \Delta/4$ for all $i \in \{q_1, q_2\} \cup \widehat{\mathcal{S}}_{\text{middle}}$, we have that, for all $i \in \{q_1, q_2\} \cup \widehat{\mathcal{S}}_{\text{middle}}$,

$$\begin{aligned}\gamma &< \widehat{d}_{q_1} + \alpha_{q_1}\\ &\le \widehat{d}_i + \alpha_{q_1}\\ &\le d_i + \alpha_i + \alpha_{q_1}\\ &\le d_i + \frac{\Delta}{2}.\end{aligned} \tag{22}$$

We also have for all $j \in \widehat{\mathcal{S}}_{\text{far}}$ that

$$\begin{aligned}\gamma &< \widehat{d}_{q_1} + \alpha_{q_1}\\ &\le \widehat{d}_{q_2} - \alpha_{q_2} + \alpha_{q_2} + \alpha_{q_1}\end{aligned}$$

$$
\begin{aligned}
&\leq \widehat{d}_j - \alpha_j + \alpha_{q_2} + \alpha_{q_1} \\
&\leq d_j + \alpha_{q_2} + \alpha_{q_1} \\
&\leq d_j + \frac{\Delta}{2}. \qquad (23)
\end{aligned}
$$

Combining (22) and (23), we have that $d_i > d_k$ for all $i \in \{q_1\} \cup \widehat{\mathcal{S}}_{\text{middle}} \cup \widehat{\mathcal{S}}_{\text{far}}$, which is a contradiction, since at most $n-k$ values of i can satisfy this inequality.

The case that $\gamma > \widehat{d}_{q_2} - \alpha_{q_2}$ is entirely analogous.

v. When $q_1 > k+h$ or $b_2 \leq k$, we can make similar arguments to the previous cases to reach a contradiction.

A.2 Proof of Fact 5

First, when $i \leq k$, we have

$$
\begin{aligned}
\widehat{d}_i + 3\alpha_i &\leq d_i + 4\alpha_i \\
&\leq d_i + \frac{\Delta_i}{2} \\
&\leq \frac{d_{k+1+h} + d_i}{2} \leq \gamma, \qquad (24)
\end{aligned}
$$

where the last inequality uses $d_i \leq d_k$, so $\mathcal{E}_{\text{bad}}(i)$ does not occur. This is similarly shown for $i > k+h$. For $k < i \leq k+h$, that $\mathcal{E}_{\text{bad}}(i)$ does not occur follows immediately from $\alpha_i \leq \Delta_i/8 \leq \Delta_i/4$.

A.3 Proof of Fact 6

Recalling that $\alpha_i(u) = \sqrt{\frac{2\beta(u,\delta/n)}{u}}$, at $\alpha_i(u) = \Delta_i/8$ we have that $u = 2(\Delta_i/8)^{-2}\beta(u,\delta')$, so we need to bound the greatest fixed point u^* of

$$
f(u) = 2(\Delta_i/8)^{-2}\beta(u,\delta').
$$

Let $u_0 = 2(\Delta_i/8)^{-2}$, and note that for all $u \geq u_0$,

$$
\begin{aligned}
f'(u) &= \frac{2(\Delta_i/8)^{-2}(2)}{u\log(1.12u)} \\
&\leq \frac{2(\Delta_i/8)^{-2}(2)}{2(\Delta_i/8)^{-2}\log((1.12)2(\Delta_i/8)^{-2})} \\
&\leq \frac{2}{\log((1.12)32)} \\
&< 1. \qquad (25)
\end{aligned}
$$

The second inequality holds because $\Delta_i \leq 2$. Suppose that $u^* > u_0$. Using Taylor's theorem, we have that for some $z \geq u_0$,

$$
\begin{aligned}
f(u_0) &= f(u^*) + f'(z)(u_0 - u^*) \\
&= u^*(1 - f'(z)) + u_0 f'(z). \qquad (26)
\end{aligned}
$$

Then

$$
\begin{aligned}
u^* &= \frac{f(u_0) - u_0 f'(z)}{1 - f'(z)} \\
&\leq \frac{f(u_0)}{1 - f'(u_0)}. \qquad (27)
\end{aligned}
$$

So, we can bound the greatest fixed point of f as

$$
\begin{aligned}
u^* &\leq \max\left\{u_0, \frac{f(u_0)}{1 - f'(u_0)}\right\} \\
&= 2(\Delta_i/8)^{-2}\max\left\{1, \frac{\beta(2(\Delta_i/8)^{-2},\delta')}{1 - 2/\log((1.12)32)}\right\} \\
&= c_1\Delta_i^{-2}\beta(2(\Delta_i/8)^{-2},\delta'), \qquad (28)
\end{aligned}
$$

where $c_1 = 128/(1 - 2/\log((1.12)32))$. Since $\widetilde{T}_i \leq u^* + 1$, letting $c_2 = c_1 + 1$,

$$
\widetilde{T}_i \leq \frac{c_2}{\Delta_i^2}\log\left(125\frac{n}{\delta}\log\left(\frac{(1.12)128}{\Delta_i^2}\right)\right). \qquad (29)
$$

Then for c sufficiently large,

$$
\widetilde{T}_i \leq c\log\left(\frac{n}{\delta}\right)\frac{\log(2\log(2/\Delta_i))}{\Delta_i^2}. \qquad (30)
$$

B ADDITIONAL THEOREM 2 PROOF DETAILS

In this section we provide details on bounding $\sum_{i\in\mathcal{S}_1(\nu)} \mathbb{E}_\nu[N_i]$ that we omitted in the proof of Theorem 2. We consider the set $\mathcal{M} = \{\ell_1, \ldots, \ell_{h+1}\} \subseteq \mathcal{S}_1(\nu)$ and construct an alternative distribution ν' such that under that distribution $\mathcal{M} \subseteq \mathcal{S}_2(\nu')$. Then under ν', if $\mathcal{A}$ succeeds, then at most h elements of $\mathcal{S}_2(\nu')$ can be in $\widehat{\mathcal{S}}$, meaning that at least one element of $\mathcal{M}$ is not in $\widehat{\mathcal{S}}$ and that $\mathcal{E}$ does not occur. So, if $\mathcal{A}$ succeeds with probability at least $1-\delta$, then both $\mathrm{P}_\nu[\mathcal{E}] \geq 1-\delta$ and $\mathrm{P}_{\nu'}[\mathcal{E}] \leq \delta$.

Our alternative distribution ν' is defined as

$$
\nu'_i = \begin{cases} \nu_{k+1+h}, & i \in \mathcal{M} \\ \nu_i, & \text{otherwise.} \end{cases}
$$

Again, to avoid ties, for $\ell \in \mathcal{M}$, one should take $\nu'_\ell = \nu_{k+1+h} + \varepsilon$ and let $\varepsilon \to 0$, but we omit this detail. The remainder of the arguments are entirely analogous to the case shown previously, giving us the bound

$$
\sum_{i\in\mathcal{S}_1(\nu)} \mathbb{E}_\nu[N_i] \geq \log\frac{1}{2\delta}\sum_{i=1}^{k-h}\frac{d_{k+1+h}(1 - d_{k+1+h})}{(d_i - d_{k+1+h})^2}.
$$